Standard Auction Revisited
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Abstract

This paper tries to find out a necessary and sufficient condition for the existence of a symmetric and increasing equilibrium bidding strategy for buyers in a standard auction. The paper considers a standard auction with one seller (whose valuation of the object is zero) and shows that an increasing bidding strategy constitutes a symmetric Bayesian Nash equilibrium if and only if the bidding strategy, derived from the first order condition of expected payoff maximization, is increasing in valuations. We also provide two examples to illustrate our results.

Keywords: Auctions, Efficiency, Independent Private Valuation

Introduction:

One of the earliest breakthroughs in Auction Theory came in 1960, with W. Vickrey's paper. After that, many well known economists and mathematicians worked in this field, and made auction theory an essential tool for many governmental and non-governmental decision-making. In 1993-94, there were radio spectrum auctions in the United States, which started applying the theory in practice.

Among the most commonly known auction literature, Milgrom (1985) provides a logical and convincing account of the theory of symmetric single object auction and also suggests how the theory can be extended in cases of multiple objects, where the bidders have single unit demand. McAfee and McMillan (1987) have also focused on the symmetric single object auctions but emphasize many extensions and applications of the theory.

This paper will analyze standard auctions. By standard auction we mean that the rule of the auction is such that the highest bidder will always win the auction. This format is very common in theory as well as in practice. Many familiar auction formats fall in this category viz, first price sealed bid auction, second price sealed bid auction, all pay auction, sad loser auction, Santa Clause auction etc.

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We will focus on the symmetric, increasing and continuous bidding strategies. As in the case of other standard auction literature, here also the idea is, first to assume a monotonically increasing bid function for each bidder, so that in our analysis we can safely take the inverse of the bid function (monotonicity guarantees that the inverse of the function is also a function and not a correspondence). After that, we will derive the bid function, and we will derive a sufficient condition that will ensure the existence of an equilibrium bidding strategy which is a monotonically increasing function of the valuation. The resulting equilibrium is always ex-post efficient in absence of reserve price and entry fee.

The main assumptions are stated below. We retain all the assumptions of standard auction along with the possibility of positive reserve price and entry fee.

**Notations and Assumptions:**
- There are $N$ buyers where $N > 0$ and a single seller.
- The seller has an indivisible object.
- For simplicity we assume that the valuation of the seller is zero.
- Valuation of the $i^{th}$ buyer is $V_i$ for all $i = 1, ..., N$.
- All the buyers' valuations lie in the interval $[0, \omega]$ i.e. $\forall j = 1, ..., N V_j \in [0, \omega]$. The valuations are distributed with a distribution function $F(.)$ which is increasing. $F(.)$ admits a continuous density $f(.) \equiv F'(.)$ and has full support.
- All the buyers are risk neutral i.e. they want to maximize their respective expected payoffs.
- Buyers are not subject to any liquidity constraint.
- All components of the model other than the valuations are assumed to be commonly known to all the bidders and the seller.
- This is a single stage game where each bidder has to submit a sealed bid.
- The highest bidder wins the auction and gets the object.
- The payment function of the $j^{th}$ highest bidder is $M_j(b)$ for all $j = 1, ..., N$ where $b$ is the bid of the $j^{th}$ highest bidder.
- Assume that there is a reserve price $R \geq 0$ and an entry fee $e \geq 0$.

**The Model:**

We now derive the symmetric, increasing equilibrium bidding strategy of this auction (if one exists) and we will try to find out the necessary and sufficient condition of the existence of such an equilibrium.

Payoff of buyer(bidder) $i$ is given below, when the bid is $b_i$
\[
\Pi_i = \begin{cases} 
v_i - M_i(b_i) - e & \text{if } b_i > \max_{j \neq i} b_j \geq R \\
-M_i(b_i) - e & \text{if } R \leq b_i < \max_{j \neq i} b_j
\end{cases}
\]

If there is more than one highest bidder then each bidder will get the object with equal probability. Suppose that bidders \( j \neq i \) follow the symmetric, increasing and differentiable equilibrium strategy \( \beta^j \equiv \beta \). Suppose bidder \( i \) bids \( b_i \). We wish to determine the optimal \( \beta \).

First, notice that it can never be optimal to choose a bid \( b > \beta(\omega) \) since in that case, bidder \( i \) would win for sure and could do better by reducing his bid slightly so that he still wins with certainty but pays less. So we only need to consider those bids for which \( b \leq \beta(\omega) \). Secondly no bidder will enter into the auction unless his expected payoff from this auction is at-least zero. Let us now introduce one notation that we are going to use frequently in this chapter.

Let us define \( \Gamma_i(a) = \left( \begin{array}{c} N-1 \\ j-1 \end{array} \right) F(a)^{(N-j)}(1-F(a))^{(j-1)} \) as the probability that the \( j^{th} \) bidder is the \( j^{th} \) highest bidder if he submits a bid \( \beta(a) \). Note that the density function corresponding to \( \Gamma_i(a) \) is given by

\[
g_i(a) = G_i(a) = \left( \begin{array}{c} N-1 \\ j-1 \end{array} \right) \left[ (N-j)F(a)^{(N-j-1)}(1-F(a))^{(j-1)}f(a) - (j-1)F(a)^{(N-j)}(1-F(a))^{(j-2)}f(a) \right]
\]

Bidder \( i \) wins the auction whenever he submits the highest bid – that is, whenever \( \max_{j \neq i} \beta(V_i) < b_i \). Since at equilibrium \( \beta \) is an increasing function, \( \max_{j \neq i} \beta(V_i) = \beta(\max_{j \neq i} V_i) = \beta(Y_1) \), where \( Y_1 \equiv Y_1^{(N-1)} \), the highest of \( N-1 \) values. Bidder \( i \) wins whenever \( \beta(Y_1) < b_i \) or equivalently, whenever \( Y_1 < \beta^{-1}(b_i) \). His expected payoff is therefore

\[
EP_i(b_i) = (V_i - M_i(b_i))G_i(\beta^{-1}(b_i)) - \sum_{j=2}^{N} M_i(b_i)G_i(\beta^{-1}(b_i)) - e
\]

\[= V_iG_i(\beta^{-1}(b_i)) - \sum_{j=1}^{N} M_i(b_i)G_i(\beta^{-1}(b_i)) - e \tag{1}\]

As we are only interested in those equilibria where \( b_i \) is an increasing function of \( V_i \), therefore if we can show that \( EP_i(b_i(V_i)) = 0 \) has a unique solution in \( V_i \) then no bidder whose valuation is less than \( V_i \) will participate in this auction. Let us call that valuation \( V_c \), i.e. a bidder with valuation \( V \) will participate in this auction only if \( V \geq V_c \). We will show that there is indeed a unique solution of the equation \( EP_i(b_i(V_i)) = 0 \). Therefore, we have \( \beta(V_c) = R, \) as \( \beta \) is increasing and continuous.

For the time being let us concentrate on the equilibrium bidding strategy of this auction. We are going to derive the first order necessary condition of expected payoff maximization. From the first order condition of maximizing the equation (1) with respect to \( b_i \), yields the following equation\(^3\):

\(^3\)The detailed derivation is given in Appendix 1.
\[ g_1^i(\beta^{-1}(b_i))V_i = \sum_{j=1}^{N} M_i^j(b_i)g_1^j(\beta^{-1}(b_i)) + \beta'(\beta^{-1}(b_i)) \sum_{j=1}^{N} M_i^j(b_i)G_i^j(\beta^{-1}(b_i)) \] 

We know that at a symmetric equilibrium, \( b_i = \beta(V_i) \) and thus we have

\[ g_1^i(V_i) = \sum_{j=1}^{N} M_i^j(\beta(V_i))g_1^j(V_i) + \beta'(V_i) \sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i) \]

\[ g_1^i(V_i)V_i = \frac{d}{dV_i} \sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i) \]

and as we stated earlier that \( \beta(V_c) = R \), so using this condition we have:\(^4\)

\[ V_iG_i^1(V_i) - \int_{V_c}^{V_i} G_i^1(t) \, dt = \sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i) \]  

Note that the left hand side of the above equation is independent of \( \beta \). Therefore, we have one equation and one unknown (i.e. \( \beta \)), so it is solvable. The equation 2 is nothing but the famous “Revenue Equivalence Theorem”. Note that from the above equation it is unclear whether the derived equilibrium bidding strategy is an increasing function of the valuation or not. Below we derive a sufficient condition for an increasing equilibrium bidding strategy.

Differentiating the above equation with respect to \( V_i \) and rearranging the terms, we obtain

\[ \frac{g_1^i(V_i)V_i - \sum_{j=1}^{N} M_i^j(\beta(V_i))g_1^j(V_i)}{\sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i)} = \beta'(V_i) \]  

Note that

\[ \sum_{j=1}^{N} M_i^j(\beta(V_i))g_1^j(V_i) = \sum_{j=1}^{N} (N - 1) \frac{f(V_i)}{F(V_i)} G_i^j(V_i)M_i^j(\beta(V_i)) - \sum_{j=1}^{N} (j - 1) \frac{f(V_i)}{F(V_i)(1 - F(V_i))} G_i^j(V_i)M_i^j(\beta(V_i)) \]

(The detailed derivation of the above expression is provided in Appendix 3)

Therefore substituting for \( \sum_{j=1}^{N} M_i^j(\beta(V_i))g_1^j(V_i) \) from equation 4 in equation 3 and rearranging the terms, we obtain,

\(^4\)The detailed derivation is given in Appendix 2.
existence of increasing symmetric equilibrium bidding strategy will also satisfied.

prove next that if the condition is satisfied then the second order sufficient condition for
value \( V \) this condition.

bid auction, all pay auction, Santa Clause auction, sad loser auction etc.) satisfy
long as the condition stated in proposition 1 holds.

if the other
dition is merely a necessary condition – we have not formally established that
condition also ensures efficiency (in absence of any reserve price and/or entry
large negative number and if the denominator is negative then that summation should be a sufficiently
second summation of the numerator should not be a very large negative number
that, if the denominator in the right hand side of equation 5 is positive then the
increasing

First, suppose that the above sufficient condition is satisfied, so that we have an

\[
\beta'(V_i) = \frac{g_i^1(V_i)V_i - \sum_{j=1}^{N} M_i^j(\beta(V_i))g_i^j(V_i)}{\sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i)}
\]

\[
\beta'(V_i) = \frac{A + \sum_{j=1}^{N} \frac{(j-1)f(V_i)}{F(V_i)F(V_j)} G_i^j(V_i)M_i^j(\beta(V_i))}{\sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i)}
\]

\[
\beta'(V_i) = \frac{A + \sum_{j=1}^{N} \frac{(j-1)f(V_i)}{F(V_i)F(V_j)} G_i^j(V_i)M_i^j(\beta(V_i))}{\sum_{j=1}^{N} M_i^j(\beta(V_i))G_i^j(V_i)}
\]

where,

\[
A = g_i^1(V_i)V_i - \sum_{j=1}^{N} (N-1) \frac{f(V_i)}{F(V_i)} G_i^j(V_i)M_i^j(\beta(V_i))
\]

\[
= (N-1) \frac{f(V_i)}{F(V_i)} \left[ G_i^1(V_i)V_i - \sum_{j=1}^{N} G_i^j(V_i)M_i^j(\beta(V_i)) \right] > 0 \quad \forall V_i > V_c
\]

The required condition for \( \beta'(V_i) \) to be positive is stated in remark 1 below.

**Remark 1:** The condition that ensures an increasing bidding strategy is that, if the denominator in the right hand side of equation 5 is positive then the second summation of the numerator should not be a very large negative number and if the denominator is negative then that summation should be a sufficiently large negative number\(^5\).

Note that all the commonly known standard auctions (viz. \( N^{th} \) price sealed bid auction, all pay auction, Santa Clause auction, sad loser auction etc.) satisfy this condition.

The derivation of an equilibrium \( \beta \) is only heuristic, because the above condition is merely a necessary condition – we have not formally established that if the other \( N - 1 \) bidders follow \( \beta \), then it is indeed optimal for a bidder with value \( V_i \) to bid \( \beta(V_i) \). The lemma 1 below shows that this is indeed correct as long as the condition stated in proposition 1 holds.

**Lemma 1:** If the bidding strategy stated above is increasing then that bidding strategy constitute an increasing symmetric equilibrium bidding strategy.

Suppose that all but bidder \( i \) follow the strategy \( \beta^i \equiv \beta \) given by the equation (2). We will argue that in that case it is optimal for bidder \( i \) to follow \( \beta \) as well. First, suppose that the above sufficient condition is satisfied, so that we have an increasing \( \beta \) function. Thus in equilibrium, the bidder with the highest valuation submits the highest bid and wins the auction. Therefore the above sufficiency condition also ensures efficiency (in absence of any reserve price and/or entry
\(^5\)The condition is necessary because if there exist a symmetric equilibrium bidding strategy which is increasing then \( \beta' \) must be positive. And the condition is sufficient because we will prove next that if the condition is satisfied then the second order sufficient condition for existence of increasing symmetric equilibrium bidding strategy will also satisfied.
fee). It is not optimal for the bidder to bid a $b > \beta(\omega)$. The expected payoff of bidder $i$ with valuation $V_i$, if he bids an amount $b \leq \beta(\omega)$, is calculated as follows. Denote by $z = \beta^{-1}(b)$ the value for which $b$ is the equilibrium bid – that is, $\beta(z) = b$. Then we can write bidder $i$’s expected payoff from bidding $\beta(z)$, when his value is $V_i$, as follows

$$\Pi(b, V_i) = V_i G_i^1(z) - \sum_{j=1}^{N} M_i^j(\beta(z)) G_i^j(z) - e$$

$$= (V_i - z) G_i^1(z) + \int_{V_i}^{z} G_i^1(t) dt$$

(The detailed derivation of the final expression is given in Appendix 4)

We thus obtain that

$$\Pi(\beta(V_i), V_i) - \Pi(\beta(z), V_i) = \int_{V_i}^{V_i} G_i^1(t) dt - (V_i - z) G_i^1(z) - \int_{V_i}^{z} G_i^1(t) dt$$

$$= \int_{z}^{V_i} G_i^1(t) dt - (V_i - z) G_i^1(z) > 0$$

regardless of whether $z \geq V_i$ or $z \leq V_i$.

The preceding argument shows that bidding an amount $\beta(z') > \beta(z)$ rather than $\beta(z)$ results in a loss equal to the shaded area to the right of $X$ in Figure 1 below; similarly, bidding an amount $\beta(z') > \beta(z)$ results in a loss equal to the area to its left.

We have thus argued that if all the other bidders are following the strategy $\beta$, a bidder with a valuation $V_i$ cannot benefit by bidding anything other than $\beta(V_i)$; and this implies that $\beta$ is a symmetric equilibrium strategy.

Finally, we will show that the equilibrium payoff function is a strictly increasing function of valuation so that if there is a valuation, $V_c \geq 0$ such that $EP_i(V_c, b_i(V_c)) = 0 \Rightarrow EP_i(V_c) = 0$ then the valuation is unique because $\forall V < V_c, EP_i(b_i(V)) < 0$ and $\forall V > V_c, EP_i(b_i(V)) > 0$. Therefore, at equilibrium $\beta(V_c) = R$.

Note that the equilibrium bidding function is given by the equation below

$$\sum_{j=1}^{N} M_i^j(R) G_i^j(V_c) + \int_{V_c}^{V_i} g_i^1(t) dt = \sum_{j=1}^{N} M_i^j(\beta(V_i)) G_i^j(V_i)$$

Substituting for $\sum_{j=1}^{N} M_i^j(\beta(V_i)) G_i^j(V_i)$ in the payoff function we get
Figure 1: Losses from over/under bidding in the auction.

\[ EP_i(V_i) = V_i G_i^1(V_i) - \sum_{j=1}^{N} M_j^1(\beta(V_i)) G_j^1(V_i) - c \]

\[ = \int_{V_c}^{V_i} G_i^1(t) dt \]

(The detailed derivation of the final expression in equation 6 is provided in Appendix 5)

Note that \( EP_i(V_i) = G_i^1(V_i) > 0 \) for all \( V_i > 0 \), therefore we have proved that the equilibrium payoff function is a strictly increasing function of valuation (see Figure 2 below).

The following remark shows that the converse of the lemma 1 is also true.

**Remark 2:** If there exists an increasing, symmetric Bayesian Nash equilibrium bidding strategy then the bidding strategy derived from the first order condition of expected payoff maximization is also increasing.

The above remark directly follows from the definition of increasing, symmetric Bayesian Nash equilibrium. Thus we have proved the proposition given below

**Proposition:** An increasing bidding strategy constitutes a symmetric Bayesian Nash equilibrium if and only if the bidding strategy, derived from the first order condition of expected payoff maximization, is increasing in valuations.

Thus we have proved that the condition stated in remark 1 is both necessary and sufficient for the existence of an increasing, symmetric equilibrium bidding strategy, because that ensures the second order condition for expected payoff.
maximization problem. This implies that if in any such auction we can show that the necessary and sufficient condition we have derived above, is satisfied then there is no need to check the second order condition. Note also that, when there is neither entry fee nor reservation price, this condition also ensures efficiency if the seller has the lowest valuation. It is to be noted that the condition that we derived may not satisfy for all possible standard auctions, but in those cases no symmetric and increasing equilibrium exists. The method we used to solve this maximization problem can easily be replicated in the case of procurement auction where there are multiple sellers selling the same good but only one buyer. We will conclude this discussion with two examples (viz. First Price Sealed Bid Auction and Santa Clause Auction) as special cases of our general setup.

**First Price Sealed Bid Auction:**

In a first price sealed bid auction, each bidder submits a sealed bid $b_i$, and the winning bidder has to pay her own bid; given the bids, the payoff of bidder $i$ is given by

$$\Pi(V_i, b_i) = \begin{cases} V_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

If there are more than one bidders with the highest bid the object goes to each bidder with equal probability. Bidder $i$ wins the auction whenever he/she submits the highest bid. We also assume $R = c = 0$ i.e. there is no reserve price or entry fee, therefore we have $V_c = 0$.

Note that here $M^i_i(b_i) = b_i$ and $M^j_i(b_i) = 0 \ \forall j \geq 2$. Therefore, we have,
\[
\sum_{j=1}^{N} M_j^i (R) G_j^i (V_c) + \int_{V_c}^{V_i} g_i^1(t) dt = \sum_{j=1}^{N} M_j^i (\beta(V_i)) G_j^i (V_i)
\]
\[
\int_{0}^{V_i} g_i^1(t) dt = b_i G_i^1 (V_i)
\]
\[
b_i = \beta(V_i) = \frac{1}{G_i^1(V_i)} \int_{0}^{V_i} g_i^1(t) dt
\]

Note that \(\beta(V_i)\) is an increasing function of \(V_i\),

\[
\beta’(V_i) = g_i^1(V_i) V_i - \frac{g_i^1(V_i)}{G_i^1(V_i)^2} \int_{0}^{V_i} g_i^1(t) dt
\]
\[
\beta’(V_i) = \frac{g_i^1(V_i)}{G_i^1(V_i)^2} \left[ V_i G_i^1(V_i) - \int_{0}^{V_i} g_i^1(t) dt \right]
\]
\[
\beta’(V_i) = \frac{g_i^1(V_i)}{G_i^1(V_i)^2} \int_{0}^{V_i} G_i^1(t) dt > 0
\]

Therefore for any \(V_i > 0\) we have \(\beta’(V_i) > 0\). As we stated earlier if we can find an increasing symmetric bidding strategy from the equation (3) then we have no need to check the second order condition for expected payoff maximization. Therefore, the symmetric increasing equilibrium bidding strategy in a first price sealed bid auction is given by \(\beta(V_i) = \frac{1}{G_i^1(V_i)} \int_{0}^{V_i} g_i^1(t) dt\).

Santa Clause Auction:

In a Santa Clause auction, each bidder submits a sealed bid of \(b_i\). Each bidder \(i\) receives an amount \(\int_{0}^{b_i} G_i^1(t) dt\) and the winning bidder has to pay her bid. Given these bids, the payoff of bidder \(i\) is

\[
\Pi(V_i, b_i) = \begin{cases} 
V_i - b_i + \int_{0}^{b_i} G_i^1(t) dt & \text{if } b_i > \max_{j \neq i} b_j \\
\int_{0}^{b_i} G_i^1(t) dt & \text{if } b_i < \max_{j \neq i} b_j
\end{cases}
\]

If there are more than one bidders with the highest bid, the object goes to each bidder with equal probability. Bidder \(i\) wins the auction whenever he/she submits the highest bid. We also assume \(R = e = 0\) i.e. there is no reserve price or entry fee, therefore we have \(V_c = 0\).

Note that here \(M_i^i(b_i) = b_i - \int_{0}^{b_i} G_i^1(t) dt\) and \(M_j^i(b_i) = -\int_{0}^{b_i} G_i^1(t) dt \ \forall j \geq 2\). Therefore we have
\[ \sum_{j=1}^{N} M_j^i (R) G_j^i (V_i) + \int_{V_i}^{V_i} g_1^i (t) \, dt = \sum_{j=1}^{N} M_j^i (\beta(V_i)) G_j^i (V_i) \]

\[ \int_{0}^{V_i} g_1^i (t) \, dt = b_i G_1^i (V_i) - \int_{0}^{b_i} G_1^i (t) \, dt \]

\[ V_i G_1^i (V_i) - \int_{0}^{V_i} G_1^i (t) \, dt = b_i G_1^i (V_i) - \int_{0}^{b_i} G_1^i (t) \, dt \]

\[ (V_i - b_i) G_1^i (V_i) - \int_{b_i}^{V_i} G_1^i (t) \, dt = 0 \]

\[ b_i = V_i \]

Note that \( \beta(V_i) \) is an increasing function of \( V_i \), in particular \( \beta'(V_i) = 1 \). Again we don’t have to check the second order condition. Therefore, the symmetric increasing bidding strategy in Santa Clause auction is given by \( \beta(V_i) = V_i \).

**Conclusion:**
In this paper we have studied standard auction in its most general format. We have tried to find out a necessary and sufficient condition for the existence of an increasing and symmetric equilibrium bidding strategies of buyers, under the assumption that the valuation of the seller is zero. We proved that an increasing bidding strategy constitutes a symmetric Bayesian Nash equilibrium if and only if the bidding strategy, derived from the first order condition of expected payoff maximization, is increasing in valuations. If there are no reserve prices and entry fees then this equilibrium bidding strategy guarantees ex post efficiency.
References:


Appendix 1:

\[ V_i \frac{g_i^1(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))} = \sum_{j=1}^{N} \left[ M_j^i(b_i) \frac{g_j^1(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))} + M_j^i(b_i) G_j^i(\beta^{-1}(b_i)) \right] = 0 \]

\[ \frac{1}{\beta'(\beta^{-1}(b_i))} \left[ g_i^1(\beta^{-1}(b_i))V_i - \sum_{j=1}^{N} M_j^i(b_i) g_j^1(\beta^{-1}(b_i)) \right] = \sum_{j=1}^{N} \left[ M_j^i(b_i) G_j^i(\beta^{-1}(b_i)) \right] \]

\[ g_i^1(\beta^{-1}(b_i))V_i = \sum_{j=1}^{N} M_j^i(b_i) g_j^1(\beta^{-1}(b_i)) + \beta'(\beta^{-1}(b_i)) \sum_{j=1}^{N} \left[ M_j^i(b_i) G_j^i(\beta^{-1}(b_i)) \right] \]

Appendix 2:

\[ \int_{V_e}^{V_i} g_i^1(t) dt = \int_{V_e}^{V_i} d \left[ \sum_{j=1}^{N} M_j^i(\beta(t)) G_j^i(t) \right] \]

\[ \int_{V_e}^{V_i} g_i^1(t) dt = \sum_{j=1}^{N} M_j^i(\beta(V_i)) G_j^i(V_i) - \sum_{j=1}^{N} M_j^i(\beta(V_e)) G_j^i(V_e) \]

\[ \sum_{j=1}^{N} M_j^i(R) G_j^i(V_e) + \int_{V_e}^{V_i} g_i^1(t) dt = \sum_{j=1}^{N} M_j^i(\beta(V_e)) G_j^i(V_i) \]

\[ V_i G_i^1(V_i) - \int_{V_e}^{V_i} G_i^1(t) dt - \left[ V_e G_i^1(V_e) - \sum_{j=1}^{N} M_j^i(R) G_j^i(V_e) \right] = \sum_{j=1}^{N} M_j^i(\beta(V_e)) G_j^i(V_i) \]

\[ V_i G_i^1(V_i) - \int_{V_e}^{V_i} G_i^1(t) dt - EP_1(V_e) - e = \sum_{j=1}^{N} M_j^i(\beta(V_e)) G_j^i(V_i) \]

\[ V_i G_i^1(V_i) - \int_{V_e}^{V_i} G_i^1(t) dt - e = \sum_{j=1}^{N} M_j^i(\beta(V_e)) G_j^i(V_i) \]
Appendix 3:
\[\sum_{j=1}^{N} M_j^i(\beta(V_i))g_j^i(V_i)\]
\[= \sum_{j=1}^{N} M_j^i(\beta(V_i)) \left[ \left( \frac{N-1}{j-1} \right) \left( (N-j)F(V_i)^{(N-j-1)}(1-F(V_i))^{(j-1)}f(V_i) - (j-1)F(V_i)^{(N-j)}(1-F(V_i))^{(j-2)}f(V_i) \right) \right] \]
\[= \sum_{j=1}^{N} M_j^i(\beta(V_i)) \left[ \frac{N-1}{j-1} \right] (N-j)F(V_i)^{(N-j-1)}(1-F(V_i))^{(j-1)}f(V_i) \]
\[= \sum_{j=1}^{N} M_j^i(\beta(V_i)) \left[ \frac{N-1}{j-1} \right] (j-1)F(V_i)^{(N-j)}(1-F(V_i))^{(j-2)}f(V_i) \]
\[= \sum_{j=1}^{N} \frac{(N-j)}{F(V_i)} G_j^i(V_i)M_j^i(\beta(V_i)) - \sum_{j=1}^{N} (j-1) \frac{f(V_i)}{(1-F(V_i))} G_j^i(V_i)M_j^i(\beta(V_i)) \]
\[= \sum_{j=1}^{N} \frac{(N-1-j+1)}{F(V_i)} G_j^i(V_i)M_j^i(\beta(V_i)) - \sum_{j=1}^{N} (j-1) \frac{f(V_i)}{(1-F(V_i))} G_j^i(V_i)M_j^i(\beta(V_i)) \]
\[= \sum_{j=1}^{N} \frac{(N-1)}{F(V_i)} G_j^i(V_i)M_j^i(\beta(V_i)) - \sum_{j=1}^{N} (j-1) \frac{f(V_i)}{(1-F(V_i))} G_j^i(V_i)M_j^i(\beta(V_i)) \]

Appendix 4:
\[V_iG_1^i(z) - \sum_{j=1}^{N} M_j^i(\beta(z))G_j^i(z) - e\]
\[= V_iG_1^i(z) - \int_{V_c}^{z} g_1^i(t)dt - \sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) - e\]
\[= (V_i - z)G_1^i(z) + \int_{V_c}^{z} G_1^i(t)dt + V_cG_1^i(V_c) - \sum_{j=1}^{N} M_j^i(R)G_j^i(V_c) - e\]
\[= (V_i - z)G_1^i(z) + \int_{V_c}^{z} G_1^i(t)dt + EP_i(V_c)\]
\[= (V_i - z)G_1^i(z) + \int_{V_c}^{z} G_1^i(t)dt\]
Appendix 5:

\[
V_i G_1^i(V_i) - \sum_{j=1}^{N} M_j^i(\beta(V_i)) G_j^i(V_i) - e
\]

\[
= V_i G_1^i(V_i) - \left[ \sum_{j=1}^{N} M_j^i(R) G_j^i(V_c) + \int_{V_c}^{V_i} g_j^i(t) dt \right] - e
\]

\[
= V_i G_1^i(V_i) - \left[ V_i G_1^i(V_i) - V_c G_1^i(V_c) \right] - \left[ \sum_{j=1}^{N} M_j^i(R) G_j^i(V_c) - \int_{V_c}^{V_i} G_1^i(t) dt \right] - e
\]

\[
= \int_{V_c}^{V_i} G_1^i(t) dt + V_c G_1^i(V_c) - \sum_{j=1}^{N} M_j^i(R) G_j^i(V_c) - e
\]

\[
= \int_{V_c}^{V_i} G_1^i(t) dt + EP_i(V_c)
\]

\[
= \int_{V_c}^{V_i} G_1^i(t) dt
\]