Single vs Simultaneous Contests
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Abstract

We have studied simultaneous contests vis-a-vis a single contest. We have shown that “generally” it is better for a contest designer to design a single contest than simultaneous contests, when the cost and the performance functions of all the contestants are linear in effort. We have provided a case where abilities are uniformly distributed to strengthen the argument.

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1 Introduction:

Contest is a widely known phenomenon. In a contest the contestants put in efforts to win prizes. The contestants have to bear some costs of effort irrespective of winning or losing. In this respect it is very similar to All-Pay Auctions. The main difference between Contest and All-Pay Auction is that the payment (or the cost) function in an All-Pay Auction is determined by the Auctioneer (or the Mechanism Designer), when he declares the rules of the Auction, whereas, the cost function of a Contest is a gift of “Mother Nature” (it is given to the contestants before the contest actually begins). Contest is also widely used in many different fields. Apart from Auction theory, Contest is used in the area of Tournaments, Rent Seeking and Lobbying etc.

Contests can be classified into two broad categories viz. single contest and multiple contests. Several papers study single contest. Moldovanu and Sela (2001) study multi-prize contests under different cost structures. Glazer and Hassin (1998) have studied the multi-prize complete-information game. In this note we are interested in studying simultaneous contests. Here we will be going to compare a single contest with simultaneous contests. We will follow more or less same structure as given in Moldovanu and Sela (2001) and we will modify it as and when required.

By simultaneous contests we mean that two or more contests are happening simultaneously; therefore, a contestant can participate in one (and only one) contest. Our first objective here is to find out the equilibrium bidding strategies of the contestants and the strategy of the contest designer for fulfilling her objective. Then we compare a single contest and multiple simultaneous contests. We will derive a condition which will ensure that a single contest is better than multiple simultaneous contests. We will show that in
a situation, where abilities are distributed uniformly, the condition will be satisfied and therefore it is better to design a single contest than multiple simultaneous contests, i.e. the sum of expected performances will be higher in case of a single contest than multiple simultaneous contests.

2 Single vs Simultaneous Contests

We describe our model formally below.

2.1 Assumptions:

- Consider a contest where \( p > 0 \) prizes are awarded. The value of the \( j^{th} \) prize is \( V_j \), where \( V_1 \geq V_2 \geq \ldots \geq V_p \geq 0 \) which are common knowledge to all the agents.

- There are \( k \) contestants. The set of contestants is \( K = \{1, \ldots, k\} \).

- Without loss of generality we can assume that \( k \geq p \) (i.e., there are at least as many contestants as the number of prizes).

- The ability of the contestant \( i \) (\( a_i \)) is private information to \( i \). Abilities are drawn independent of each other from an interval \([m, 1]\) (where \( 0 < m < 1 \)) according to the distribution function \( F \), which is common knowledge. We assume that \( F \) has a continuous density \( f \) and has full support.

- In the contest each player \( i \) makes an effort \( x_i \). Efforts are undertaken simultaneously.
• Each contestant $i$ chooses her effort in order to maximize her expected utility (given the other competitors’ efforts and given the values of the different prizes). Alternatively, each contestant $i$ chooses her performance to maximize expected utility as $a_i$ is known to $i$.

• The contestant with the highest performance wins the first prize $V_1$ while the contestant with second highest performance wins the second prize $V_2$ and so on until all the prizes are allocated. That is, the payoff of contestant $i$ who has ability $c_i$, and makes an effort $x_i$ is either $V_j - \frac{1}{c_i}x_i$ if $i$ wins prize $j$, or $-\frac{1}{c_i}x_i$ if $i$ does not win a prize.

• The contest designer determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximize the expected value of the sum of the performances, $\sum_{i=1}^{k} \phi_i$ (given the contestants’ equilibrium performance function).

• The contest designer divides the $K$ contestants into two parts. The first part consists of $\alpha_K K$ number of contestants (we are assuming that $\alpha_K$ is chosen in such a way that $\alpha_K K$ becomes a positive integer), and the second part consists of $(1 - \alpha_K)K$ number of contestants. We have $0 < \alpha_K < 1$.

• The contest designer divides the $M$ amount of money into two parts. The first part consists of $\alpha_M M$ and the second part consists of $(1 - \alpha_M)M$. We have $0 < \alpha_M < 1$.

• We have already assumed that the cost functions of all the contestants are linear in efforts i.e. $Cost(x_i) = \frac{x_i}{a_i}$. Therefore, in case of the first
If a contestant wins a prize, her payoff is \( \alpha M - \frac{x_i}{a_i} \) and if she doesn’t win, then her payoff is \( -\frac{x_i}{a_i} \). In case of the second contest, if a contestant wins a prize, her payoff is \( (1 - \alpha M)M - \frac{x_i}{a_i} \) and if she doesn’t win, then her payoff is \( -\frac{x_i}{a_i} \).

Let us assume that at equilibrium, the contestant whose ability is \( m \) makes zero effort, i.e. at equilibrium we have \( x_i(m) = 0 \) \( \forall i = 1, ..., K \). Note that the equilibrium effort is a function of a contestant’s ability i.e. \( x_i = b(a_i) \), at equilibrium. We are interested in a strictly increasing bidding equilibrium, therefore \( b(.) \) is a strictly increasing function of ability at equilibrium.

As both the cost and the performance functions are linear in effort, it is optimal for the contest designer is to design a winner-take-all (single first prize) contest\(^2\).

Let us now calculate the expected payoff of contestant \( i \). If contestant \( i \) is competing in the first contest then her expected payoff is

\[
\alpha_M MF(b^{-1}(x_i))^{(\alpha K - 1)}_a - \frac{x_i}{a_i}
\]

Her objective is to maximize her expected payoff by choosing \( x_i \). Let us now derive the equilibrium bidding strategy for this contest.

**Proposition 2.1.** In a symmetric equilibrium the performance function of each contestant is given by

\[
b(c) = \alpha_M M(\alpha K - 1) \int_m^c a F(a)^{(\alpha K - 2)} F'(a) da
\]

\(^2\)See Moldovanu and Sela 2001
where $c \in [m, 1]$.

**Proof.** See appendix A.

The next proposition derives a condition which will ensure that a single contest gives a higher sum of expected performances than two simultaneous contests.

**Proposition 2.2.** It is better to design a single contest than two simultaneous contests if the inequality below holds

\[
(K - 1) \int_m^c aF(a)^{(K-2)} F'(a) da F'(c) dc \\
> \alpha_K \alpha_M (\alpha_K K - 1) \int_m^c aF(a)^{\alpha_K K - 2} F'(a) da F'(c) dc \\
\hspace{1cm} + (1 - \alpha_K)(1 - \alpha_M)((1 - \alpha_K)K - 1) \int_m^c aF(a)^{(1-\alpha_K)K-2} F'(a) da F'(c) dc \\
\forall 0 < \alpha_K < 1 \text{ and } 0 < \alpha_M < 1
\]

(2.1)

**Proof.** See appendix B.

Note that $M(K - 1) \int_m^c aF(a)^{(K-2)} F'(a) da$ is the performance of a single contestant in case of the single contest. So, $M(K - 1) \int_m^c aF(a)^{(K-2)} F'(a) da F'(c) dc$ is the expected performance of that contestant. Therefore, the aggregate expected performance is given by $K(K-1)M \int_m^c aF(a)^{(K-2)} F'(a) da F'(c) dc$.

Similarly, $\alpha_K \alpha_M (\alpha_K K - 1) \int_m^c aF(a)^{\alpha_K K - 2} F'(a) da F'(c) dc$ is the aggregate expected performance in the first contest of the two simultaneous contests and $(1-\alpha_K)(1-\alpha_M)((1-\alpha_K)K - 1)M \int_m^c aF(a)^{(1-\alpha_K)K-2} F'(a) da F'(c) dc$ is the aggregate expected performance in the second contest of the two simultaneous contests.
The next proposition states that the absolute amount of prize money is not the determining factor of whether a contest designer should design simultaneous contests than a single contest.

**Proposition 2.3.** Whether $\Phi_1 \gtrless \Phi_2$ does not depend on the total amount of money that the contest designer has for distributing as prizes.

*Proof.* Straight from equation 2.1

Note that in case of a single contest the prize money is higher than that in each of the simultaneous contests. So the expected performance of a contestant is higher in a single contest than that in case of simultaneous contests. So the sum of expected performances is also higher in a single contest. Also, the competition in a single contest is higher than that in each of the simultaneous contests. So in that way the expected performance of a contestant is higher in a single contest than that in simultaneous contests. Therefore, we would expect that “generally” a single contest is a better option for a contest designer than simultaneous contests. We are going to conclude this section by proving that the above argument is true, if we assume that the abilities of all the contestants are distributed uniformly. We already know that in this case, the distribution function is given by the equation $F(x) = \frac{x-m}{1-m}$ and the density function is given by $F'(x) = \frac{1}{1-m}$.

**Corollary 2.4.** If the abilities of all the contestants are distributed uniformly then it is always better to design a single contest rather than two simultaneous contests.

*Proof.* See appendix C.
In case of simultaneous contest the number of contestants in each contest is less than the number of contestants in the single contest. So, the competition in each contest of a simultaneous contest is less than that in the single contest. Therefore, one may expect that the aggregate effort level (and hence the aggregate performance level) in the single contest is higher than that in the simultaneous contest. Also the prize money given in each contest of a simultaneous contest is less than that in the single contest. This also implies that at equilibrium the aggregate effort level in the single contest is higher than that in the simultaneous contest. These two factors together imply that, even if one can find a case where simultaneous contest produces higher aggregate performance than the single contest, then that is due to some special properties of the corresponding distribution function. The uniform distribution is very simple and easy to use, so it is obvious that the above Corollary is true.

This result can be very easily extended for $P$ prizes where $P > 2$. In that case the equation 2.1 becomes

\[(K - 1) \int_m^c aF(a)^{(K-2)} F'(a) da F'(c) dc > \sum_{i=1}^P \alpha_{Ki} \alpha_{Mi} (\alpha_{Ki} K - 1) \int_m^c aF(a)^{\alpha_{Ki} (K-2)} F'(a) da F'(c) dc \]  

(2.2)

where $\forall i \ 0 < \alpha_{Ki} < 1$ and $\sum_{i=1}^P \alpha_{Ki} = 1$

and $\forall i \ 0 < \alpha_{Mi} < 1$ and $\sum_{i=1}^P \alpha_{Mi} = 1$.

It is routine to check that, in case of a uniform distribution, the equation 2.2 holds and therefore, it is optimal to design a single contest rather than simultaneous contests.
3 Conclusion:

We have studied simultaneous contests. We have first derived the equilibrium bidding strategies of the contestants in case of multiple simultaneous contests. Then we have compared a single contest and multiple simultaneous contests. We have derived a condition which will ensure that a single contest is better than multiple simultaneous contests. We have shown that in case, where abilities are distributed uniformly, the condition will be satisfied and therefore it is better to design a single contest than multiple simultaneous contests.

References:


Appendix A:

Let us assume \( b^{-1}(x_i) = y_i \). Therefore the first order condition of expected payoff maximization yields

\[
\alpha_M M(\alpha_K K - 1) F(y_i)^{(\alpha_K K - 2)} y_i' - \frac{1}{a_i} = 0
\]

\[
\Leftrightarrow \alpha_M M(\alpha_K K - 1) F(y_i)^{(\alpha_K K - 2)} y_i' = \frac{1}{a_i}
\]

\[
\Leftrightarrow \alpha_M M(\alpha_K K - 1) y_i F(y_i)^{(\alpha_K K - 2)} y_i' = 1
\]

\[
\Leftrightarrow \alpha_M M(\alpha_K K - 1) y_i F(y_i)^{(\alpha_K K - 2)} dy_i = d\phi_i
\]

Note that, here we are assuming an interior solution, and also that, at equilibrium \( x_i = b(a_i) \Rightarrow a_i = b^{-1}(x_i) \Rightarrow a_i = y_i(x_i) \). A contestant with the lowest possible ability \( a_i = m \) never wins a prize. Hence the optimal effort of this type is always zero. Therefore, \( y(0) = m \).

Let us define a function \( G(y) \) such that

\[
G(y) = \alpha_M M(\alpha_K K - 1) \int_m^y a F(a)^{(\alpha_K K - 2)} F'(a) da
\]

Then the solution to the differential equation above is given by \( G(y) = \phi_i = x_i \) with the boundary condition, \( y(0) = m \). Therefore, we have the following equalities \( G(y) = \phi_i \Rightarrow G(b^{-1}(\phi_i)) = \phi_i \Rightarrow G \equiv b \). Now we will show that the equilibrium performance function we have derived is indeed strictly increasing in ability. Note that

\[
b'(a_i) = \alpha_M M(\alpha_K K - 1) a_i F(a_i)^{(\alpha_K K - 2)} F'(a_i) > 0 \ \forall a_i \in (m, 1]
\]

To verify the second order condition of expected utility maximization, next we will show that
\( \Pi(x_i, a_i) = \alpha_M M F(b^{-1}(x_i))^{(\alpha_K K - 1)} - \frac{x_i}{a_i} \)

is pseudo concave, i.e. the derivative \( \Pi_x(x_i, a_i) \) is non-negative if \( x_i \) is smaller than \( b(a_i) \) and non-positive if \( x_i \) is larger than \( b(a_i) \). As \( \Pi(x_i, a_i) \) is continuous in \( x_i \) this implies that \( \Pi(x_i, a_i) \) is maximized at \( x_i = b(a_i) \).

\[ \Pi_{x_i}(x_i, a_i) = \alpha_M M(\alpha_K K - 1)F(b^{-1}(x_i))^{(\alpha_K K - 2)}F'(b^{-1}(x_i)) \frac{db^{-1}(x_i)}{dx_i} - \frac{1}{a_i} \]

Let \( x_i < b(a_i) \) and let \( \hat{c} \) be the type who is supposed to bid i.e. \( b(\hat{c}) = x_i \). Note that \( \hat{c} < a_i \) since \( b(.) \) is strictly increasing. Differentiating \( \Pi_{x_i}(x_i, a_i) \) with respect to \( a_i \) we get \( \Pi_{x_i a_i}(x_i, a_i) = \frac{1}{a_i} > 0 \) i.e. \( \Pi_{x_i}(x_i, a_i) \) is increasing in \( a_i \). Since \( \hat{c} < a_i \), we obtain \( \Pi_{x_i}(x_i, a_i) \geq \Pi_{x_i}(x_i, \hat{c}) \). Since \( b(\hat{c}) = x_i \), so \( \Pi_{x_i}(x_i, \hat{c}) = 0 \) would imply \( \Pi_{x_i}(x_i, a_i) \geq 0 \) for every \( x_i < b(a_i) \). A similar argument shows that \( \Pi_{x_i}(x_i, a_i) \leq 0 \) for every \( x_i > b(a_i) \).

**Appendix B:**

First, let us calculate the sum of the expected performances of all the contestants in this contest. The expected performance of a single contestant is given by the equation below -

\[ \int_m^1 \alpha_M M(\alpha_K K - 1) \int_m^c aF(a)^{(\alpha_K K - 2)}F'(a) da F'(c) dc \]

\[ = \alpha_M M(\alpha_K K - 1) \int_m^1 \int_m^c aF(a)^{(\alpha_K K - 2)}F'(a) da F'(c) dc \]

There are \( \alpha_K K \) number of contestants, therefore the sum of the expected performances of all the contestants is given by the equation below -

\[ \alpha_K K \alpha_M M(\alpha_K K - 1) \int_m^1 \int_m^c aF(a)^{(\alpha_K K - 2)}F'(a) da F'(c) dc \]
Now consider two different games. In the first game, \( K \) numbers of contestants are contesting for a single prize of value \( M \), whereas in the second game, two contests are going on simultaneously. In the first contest \( \alpha K \) number of contestants are contesting for a single prize of value \( M_1 = \alpha M \), and in the second contest \( (1 - \alpha K)K \) number of contestants are contesting for a single prize of value \( M_2 = (1 - \alpha M)M \). Our objective is to compare the sum of the expected performances of all the contestants in both the games. We denote the sum of the expected performances of all the contestants in the first game by \( \Phi_1 \) and that in the second game by \( \Phi_2 \). If \( \Phi_1 > \Phi_2 \) for all \( 0 < \alpha_K < 1 \) and for all \( 0 < \alpha_M < 1 \) then we can conclude that designing a single contest is better than designing two simultaneous contests. On the other hand, if \( \Phi_1 < \Phi_2 \) for all \( 0 < \alpha_K < 1 \) and for all \( 0 < \alpha_M < 1 \), then we can say that designing two simultaneous contests is better than designing a single contest.

Note that,

\[
\Phi_1 = KM(K - 1) \int_m^c \int_m^c aF(a)^{(K - 2)} F'(a) da F'(c) dc
\]

and

\[
\Phi_2 = \alpha_K K \alpha_M M(\alpha_K K - 1) \int_m^c \int_m^c aF(a)^{(\alpha_K K - 2)} F'(a) da F'(c) dc
\]

\[+(1 - \alpha_K)K(1 - \alpha_M)M((1 - \alpha_K)K - 1) \int_m^c \int_m^c aF(a)^{(1 - \alpha_K)K - 2} F'(a) da F'(c) dc\]
Therefore, we have,

\[ \Phi_1 > \Phi_2 \]

\[ \Leftrightarrow KM(K - 1) \int_m^c aF(a)(K - 2)F'(a)daF'(c)dc \]

\[ > \alpha_K \alpha_M M(\alpha_K K - 1) \int_m^c aF(a)(\alpha_K K - 2)F'(a)daF'(c)dc \]

\[ + (1 - \alpha_K)K(1 - \alpha_M)M((1 - \alpha_K)K - 1) \int_m^c aF(a)((1 - \alpha_K)K - 2)F'(a)daF'(c)dc \]

\[ \forall 0 < \alpha_K < 1 \text{ and } 0 < \alpha_M < 1 \]

\[ \Leftrightarrow (K - 1) \int_m^c aF(a)(K - 2)F'(a)daF'(c)dc \]

\[ > \alpha_K \alpha_M (\alpha_K K - 1) \int_m^c aF(a)(\alpha_K K - 2)F'(a)daF'(c)dc \]

\[ + (1 - \alpha_K)(1 - \alpha_M)((1 - \alpha_K)K - 1) \int_m^c aF(a)((1 - \alpha_K)K - 2)F'(a)daF'(c)dc \]

\[ \forall 0 < \alpha_K < 1 \text{ and } 0 < \alpha_M < 1 \]

**Appendix C:**

The sum of the expected performances in the first game is given by

\[ \Phi_1 = KM \int_m^1 (K - 1) \int_m^a yF(y)(K - 2)f(y)dyf(a)da \]

\[ = KM(K - 1) \int_m^a yF(y)(K - 2)f(y)dyf(a)da \]

\[ = KM(K - 1) \int_m^a y \left( \frac{y}{1-m} \right)^{(K-2)} \frac{1}{1-m}dy \frac{1}{1-m}da \]

\[ = M \left[ 1 - \frac{2}{K+1}(1 - m) \right] \]

Let us now calculate the sum of the expected performances of all the contestants for the second game.

\[ \Phi_2 = \alpha_M M \left[ 1 - \frac{2}{\alpha_K K+1}(1 - m) \right] + (1 - \alpha_M)M \left[ 1 - \frac{2}{(1-\alpha_K)K+1}(1 - m) \right] \]

\[ = M \left[ 1 - 2\alpha_M \frac{1}{\alpha_K K+1}(1 - m) - \frac{2(1-\alpha_M)}{(1-\alpha_K)K+1}(1 - m) \right] \]

\[ = M \left[ 1 - 2(1 - m) \left( \frac{\alpha_M}{\alpha_K K+1} + \frac{(1-\alpha_M)}{(1-\alpha_K)K+1} \right) \right] \]
Finally, we compare both of them and obtain,

\[ \Phi_1 \geq \Phi_2 \]
\[ \Leftrightarrow M \left[ 1 - \frac{2}{K+1} (1 - m) \right] \geq M \left[ 1 - 2(1 - m) \left( \frac{\alpha_M}{\alpha_K K+1} + \frac{(1-\alpha_M)}{(1-\alpha_K)K+1} \right) \right] \]
\[ \Leftrightarrow \frac{2}{K+1} (1 - m) \leq 2(1 - m) \left( \frac{\alpha_M}{\alpha_K K+1} + \frac{(1-\alpha_M)}{(1-\alpha_K)K+1} \right) \]
\[ \Leftrightarrow \frac{1}{\alpha_K K+1 - (1-\alpha_K)K+1} \leq \left( \frac{\alpha_M}{\alpha_K K+1} + \frac{(1-\alpha_M)}{(1-\alpha_K)K+1} \right) \]
\[ \Leftrightarrow 1 \leq \left( \alpha_M + \frac{\alpha_M(1-\alpha_K)K}{\alpha_K K+1} + (1 - \alpha_M) + \frac{(1-\alpha_M)\alpha_K K}{(1-\alpha_K)K+1} \right) \]
\[ \Leftrightarrow 0 \leq \left( \frac{\alpha_M(1-\alpha_K)K}{\alpha_K K+1} + \frac{(1-\alpha_M)\alpha_K K}{(1-\alpha_K)K+1} \right) \]

Note that 0 < \( \alpha_K < 1 \) and 0 < \( \alpha_M < 1 \), therefore 0 < (1 - \( \alpha_K \)) < 1 and 0 < (1 - \( \alpha_M \)) < 1.